The Light-Cone Wave Function of the Pion

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The light–cone wave function of the pion is calculated within the Nambu–Jona-Lasinio model. The result is used to derive the pion electromagnetic form factor, charge radius, structure function, π–γ–transition form factor and distribution amplitude.

1 Introduction

A light–cone (LC) wave function is a localized (i.e. normalizable) stationary solution of the LC Schrödinger equation $i\partial_\tau|\Psi(\tau)\rangle = H_{\rm LC}|\Psi(\tau)\rangle$, which describes the evolution of a state $|\Psi(\tau)\rangle$ in LC time $\tau \equiv x^+ = x^0 + x^3$, conjugate to the LC Hamiltonian $H_{\rm LC} \equiv P^- = P^0 - P^3$ [1,2]. In quantum field theory, a stationary solution $|\Psi\rangle$ has a LC Fock expansion which in principle does not terminate, as can be seen e.g. for the pion, $|\pi\rangle = \psi_2|q\bar{q}\rangle + \psi_3|q\bar{q}g\rangle + \psi_4|q\bar{q}q\bar{q}\rangle + \ldots$. The LC wave functions are the amplitudes ψ_n to find n particles with momenta p_i (quarks q, antiquarks $\bar{q}$ or gluons g) in a pion of momentum P, i.e. $\psi_n(p_1, \ldots p_n; P) \equiv \langle n|\pi\rangle = \langle p_1, \ldots, p_n|\pi(P)\rangle$. We shall see in a moment that the momentum dependence of the ψ_n is of a very peculiar nature.

Upon choosing LC rather than ordinary time one benefits from the following virtues. First of all, the field theory vacuum becomes 'trivial', in particular $P^-|0\rangle = 0$. As a result, disconnected vacuum contributions do not mix with the LC wave functions: $\langle 0|P^-|n\rangle = 0$. Unlike in ordinary 'instant–form' quantization, boosts become *kinematical* (interaction independent). This is closely related to a twodimensional Galilei invariance which leads to a *separation* of center–of–mass and relative coordinates. These features imply that the ψ_n only depend on the *frame independent* relative momenta $x_i = p_i^+/P^+$ and $\mathbf{k}_{\perp i} = x_i \mathbf{P}_\perp - \mathbf{p}_{\perp i}$.

As the number of difficulties is always conserved, the list of advantages goes along with a list of problems. There is the conceptual question how nontrivial condensates can arise in a trivial vacuum. The answer is crucial for understanding spontaneous symmetry breaking in the LC framework. Instead of boosts, rotations and parity become dynamical and thus complicated. There are also more technical problems. Due to the lack of explicit covariance and rotational invariance, renormalization becomes very difficult beyond one loop. There are simply not enough *manifest* symmetry principles providing a guideline. As a result, there is an abuncance of counterterms which can even become non–local ($\sim 1/k^+$). In addition, LC field theories are constrained systems. This makes their quantization nonstandard and somewhat involved. Finally, the LC Schrödinger equation in general represents an infinity of coupled, nonlinear integral equations for the amplitudes ψ_n. To solve them one has to resort to truncations. It should be noted that for an *ab–initio* calculation of LC wave functions basically all this problems have to be solved.

If it is so hard, then why should one calculate LC wave functions? It is worth the effort because their determination implies the knowledge of the entire hadron structure (just like the determination of the Coulomb wave functions implies the knowledge of the hydrogen structure). Unlike for nonrelativistic systems, however, there is a subtlety. The (relativistic) LC wave functions depend on a *resolution scale* Q, $\psi_n = \psi_n(Q)$. It is well known that there are two basic regimes, a 'soft' and a 'hard' one. In the hard regime, one has $Q \gg \Lambda_{\mathrm{QCD}}$ so that perturbative QCD is applicable. The LC wave functions describe the distribution of *partons*. If the resolution scale Q is of the order of Λ_{QCD}, in the soft regime, perturbation theory no longer works. Hadrons are believed to consist of effective *constituent* quarks which should again be reflected in the nature of the LC wave functions. Low–energy effective field theories seem to provide a reasonable description of the soft regime, at least as far as spontaneous chiral symmetry breaking (χSB) is concerned. Chiral quark models, in particular, with *explicit* quark degrees of freedom, should yield soft LC wave functions describing hadrons as bound states of a few constituent quarks. Such a description should be valid for $Q \lesssim 1$ GeV. Whether these expectations are true will be examined in this contribution.

2 Formalism

It turns out that most of the difficulties mentioned in the introduction can be circumvented using a particular chiral quark model originally due to Nambu and Jona-Lasinio [3]. For two flavors the Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2] \; . \qquad (1)$$

The model is not renormalizable whence it requires a cutoff Λ, the value of which is fixed by phenomenology. Beyond a critical coupling G_c one has spontaneous χSB together with dynamical mass generation, $m_0 \to m \sim \langle 0|\bar{\psi}\psi|0\rangle$. The current quarks thus become constituent quarks with a mass proportional to the chiral condensate. As a constituent picture is realized one expects a truncation of the LC Schrödinger equation to work reasonably well. We avoid to solve complicated constraint equations [4–6] by using a Schwinger–Dyson approach as pioneered by 't Hooft [7]. It consists of two steps. First one solves the Schwinger–Dyson equation for the quark self–energy Σ in mean–field approximation (equivalent to the large–N_C limit). This yields the gap equation $\Sigma = const = m \sim \langle 0|\bar{\psi}\psi|0\rangle_\Lambda$. To make sense of the quadratically divergent expression for the condensate one uses an *invariant–mass cutoff* Λ satisfying $M_0^2 \equiv (k_\perp^2 + m^2)/x(1-x) \le \Lambda^2$, $x \equiv k^+/\Lambda$ (for an alternative, see [8]). It regulates the divergence of the condensate both for $x \to 0, 1$ and $k_\perp \to \infty$, and is related to the 3–vector cutoff $\Lambda_3 \ge |\mathbf{k}|$ by $\Lambda^2 = 4(\Lambda_3^2 + m^2)$. Accordingly, the result for the condensate agrees with the usual one [2,3].

Knowing the effective quark mass m one performs the second step in the program and solves the Bethe–Salpeter equation in the pseudoscalar channel (using ladder approximation/large–N_C). The LC wave function of the pion is obtained by three–dimensional reduction (integrating over LC energy). In the chiral limit, the result is

$$\begin{pmatrix} \psi_{2\uparrow\uparrow} & \psi_{2\uparrow\downarrow} \\ \psi_{2\downarrow\uparrow} & \psi_{2\downarrow\downarrow} \end{pmatrix} = -\frac{N}{k_\perp^2 + m^2} \begin{pmatrix} -2mk_- & m^2 - k_\perp^2 \\ k_\perp^2 - m^2 & -2mk_+ \end{pmatrix} \theta(\Lambda^2 - M_0^2)\,, \qquad (2)$$

where we have defined $k_\pm \equiv k_1 \pm i k_2$ corresponding to the $L_z = \pm 1$ components of the wave function. N is a normalization constant to be determined later. The step function implements the invariant–mass–cutoff. It is only for this cutoff that the wave function becomes normalizable.

3 Results

With the pion wave function at hand one can go on and calculate observables. To make life simple I will always work in the chiral limit $m_0 = 0 = M_\pi$. Due to this limit the wave function (2) does not depend on x (apart from the cutoff). Further simplifications arise in the large–cutoff limit (LCL). This amounts to keeping only the leading order in $\epsilon^2 \equiv m^2/\Lambda^2$. As a result one can find analytic expressions for all observables [1]. The LC wave function (2) simplifies to $\psi_{2\uparrow\uparrow} =$

[1] The same logic is used in the context of the instanton model [9] where ϵ^2 corresponds to the 'packing fraction' of instantons in the vacuum.

$\psi_{2\downarrow\downarrow} = 0$ and $\psi_{2\uparrow\downarrow} = -\psi_{2\downarrow\uparrow} = N\theta(\Lambda^2 - M_0^2)$. The nontrivial components of the wave function thus just become step functions [10]. We are thus left with two parameters, N and Λ, which are determined by normalizing ψ_2 to unity (*enforcing* a constituent picture) and using the pion decay constant f_π. The latter is basically given by the wave function 'at the origin' [11]. This results in $N = \sqrt{3}/f_\pi$ and $\Lambda = 4\pi f_\pi \simeq 1.16$ GeV. The value for Λ exactly coincides with the Georgi–Manohar scale [12] below which chiral effective theories make sense. It is also consistent with the constraint on the wave function stemming from $\pi^0 \to 2\gamma$ [11],

$$\int dx\, \psi_{2\uparrow\downarrow}(x, \mathbf{0}_\perp) = \sqrt{3}/f_\pi \equiv N \; . \tag{3}$$

With all parameters determined one can use the Drell–Yan formula as discussed e.g. in [11] and calculate the pion *charge radius*. The result, $r_\pi^2 = 3/4\pi^2 f_\pi^2 \simeq (0.60 \text{ fm})^2$, agrees with those obtained in a covariant Bethe–Salpeter approach [13] and the instanton model [9]. The slight difference compared to the experimental value of 0.66 fm stems from the use of the LCL.

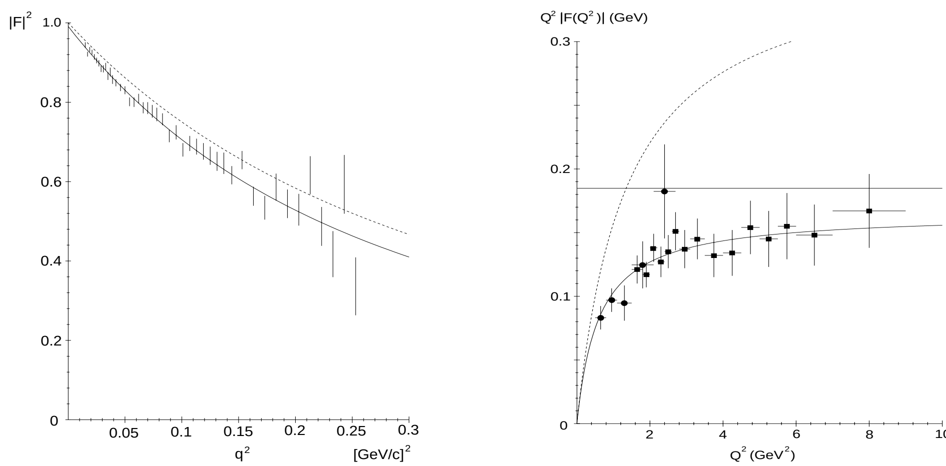

Fig. 1. *left:* pion electromagnetic form factor squared vs. momentum transfer $q^2 \equiv q_\perp^2$; dashed line: pole fit $|F|^2 = 1/(1+q^2 r_\pi^2/6)$ with $r_\pi = 0.60$ fm; full line: same formula with experimental value as extracted from the data [14]; *right:* $\pi\gamma$ transition form factor times momentum transfer Q^2; dashed line: NJL pole formula (4); full line: same formula with Λ^2 replaced by $\Lambda^2/2$; full horizontal line: asymptotic value $2f_\pi$; circles: CELLO data [15]; squares: CLEO data [16].

A pole fit to the form factor data of Amendolia et al. [14] is shown in the left–hand part of Fig. 1. The momentum space wave function (2) readily yields the r.m.s. transverse momentum, $\langle k_\perp^2 \rangle = \Lambda^2/10 \simeq (370 \text{ MeV}^2)$, which makes it obvious that the pion is a highly relativistic system.

The valence quark distribution f^v (or pion *structure function*) is given by the square of the wave function integrated over $k_\perp$ resulting in $f^v(x) = 6x(1-x)$.

This coincides with the results of [5,17] and qualitatively agrees with the empirical parton distributions of [18] if the relevant scale is defined as $Q^2 \equiv \langle x\rangle(1-\langle x\rangle)\Lambda^2 \simeq (600 \text{ MeV})^2$, the mean $\langle x\rangle$ being 1/2.

The *transition form factor* for the process $\gamma\gamma^* \to \pi^0$ can be calculated according to [11]. One finds the pole formula

$$F_{\gamma\gamma^*\pi}(Q^2) = \frac{1}{4\pi^2 f_\pi} \frac{1}{1+Q^2/\Lambda^2} , \tag{4}$$

which is displayed in the right–hand part of Fig. 1. The low–Q^2 behavior is fixed by (3) and thus fine. The behavior for $Q^2 \to \infty$ is off by a factor of two, which, however, is not bothersome as this regime is way beyond where the NJL model makes sense. One can fix the large–Q^2 behavior by introducing an effective, spin–averaged wave function [10]. This simply amounts to replacing $N \to 2N$ and $\Lambda^2 \to \Lambda^2/2$ in (2) and (4). In this case, (3) cannot be maintained and the logic based on low–energy chiral dynamics does no longer apply.

The *pion distribution amplitude* $\phi_{\rm NJL}$ is given by the $k_\perp$–integral of the wave function. As the square of a step function is again a step function the distribution amplitude coincides with f^v and hence with the *asymptotic* distribution amplitude, $\phi_{\rm NJL}(x) = 6x(1-x) \equiv \phi_{\rm as}(x)$. Recently this quantity has been measured at Fermilab [19]. At a rather low momentum scale of $Q \simeq 3$ GeV ϕ turns out to be close to asymptotic. This provides further evidence that the approach presented above makes sense.

4 Conclusions

I have analytically determined the LC wave function of the pion within a simple field theoretic model (NJL) which is known to yield a good description of χSB. With the pion wave function at hand I was able to calculate a number of observables which were accurate to within 10 %, consistent with the estimated limitations of the LCL employed. Particularly important is the result that a constituent picture does make sense: higher Fock states seem to be unimportant. If one wants to go beyond the LCL, one faces a fine–tuning problem in ϵ^2. Delicate numerical fits will thus be necessary. It would be interesting to refine the presented method by choosing more realistic models. Possibilities would be (i) the instanton vacuum [9] which is more closely related to QCD, (ii) an elaborate Schwinger–Dyson approach with more realistic kernels and propagators [20], or (iii) an effective field theory approach in terms of constituent quarks which, at the moment, is still in its infancy [12].

Acknowledgments

I cordially thank the organizers S. Bielefeld, L. Hollenberg and H.-C. Pauli for their efforts which resulted in such a stimulating meeting.

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